NEU PRESS

EDUTECH RESEARCH

Volume: 3 Issue: 2 Year: 2025 | Research Article | e-ISSN: 3023-6665

Analyzing Conceptual Understanding of Rate of Change: A Comparison of Conventional and Revised Textbooks

Hyunjeong Lee[1*]

[1] Indiana University, Bloomington, IN, USA

Article Info

Received: 11.11.2025
Accepted: 20.12.2025
Published: 31.12.2025

Keywords:
Rate of change,
Conventional textbook,
Revised textbook,
Conceptual understanding.

ABSTRACT

Rate of change is a central idea in mathematics, serving as a critical conceptual bridge from introductory algebra to the sophisticated mathematical meanings required in calculus. The initial formal introduction of this concept occurs in Algebra 1, where it is presented as the slope of a linear function. Robust understanding of slope at this stage is essential, as it integrates students' earlier knowledge of ratio and proportion within a functional context. The purpose of this paper is to conduct a comparative analysis of how the conceptual understanding of rate of change is developed in two distinct pedagogical approaches: one conventional and one standards-based Algebra 1 textbook. The central research question guiding this study is: What similarities and differences exist in the presentation of rate of change, particularly in the areas of ratio, proportion, and covariational reasoning, across these two textbooks? The methodology employs a detailed conceptual analysis, exploring the inherent mathematical meaning embedded in the instructional materials. Analytical framework focuses on three core components: (1) ratio (multiplicative comparison), (2) proportion (equality of ratios), and (3) covariational reasoning (coordination of changes between two variables). The analysis reveals that while both textbooks share certain foundational components, such as the coordination of values (covariation) and basic multiplicative comparison (ratio), significant pedagogical differences emerge. Specifically, the standards-based textbook more explicitly fosters the conceptual understanding of the iteration and partitioning of a composed unit and the exploration of the constant equality of ratios. Conversely, the conventional textbook heavily emphasizes procedural fluency and computational skills, often providing explicit formulas and embedded solutions with less demand for student justification. This paper contends that the standards-based approach effectively prioritizes and challenges students' deeper mathematical reasoning and independent exploration, suggesting different learning trajectories for foundational concepts based on curricular design.

To cite this article:

Lee, H. (2025). Analyzing conceptual understanding of rate of change: A comparison of conventional and revised textbooks. *Edutech Research, 3*(2), 53-69.

***Corresponding Author:** Hyunjeong Lee, *hle5@iu.edu*

INTRODUCTION

Rate of change is a central idea in mathematics and, in particular, supports students towards rich mathematical meanings in calculus. The first place where the rate of change typically appears in curricular materials is in Algebra 1 courses in lessons on linear functions. As a linear function is an entry point to functions, students are exposed to learning the slope (or the rate of change) of a line in the linear function chapter of many mathematics textbooks. Understanding rate of change in the context of linear functions builds on students' earlier ideas of ratio and proportion. Also, covariational reasoning, is a key component in interpreting and representing functions, generally (Carlson et al., 2002). Therefore, for secondary students to develop robust understandings of rate of change involves developing their understanding of ratio, proportion, and covariational reasoning (Thompson, 1994). Developing robust understandings of the rate of change, however, is quite challenging for secondary students.

Textbooks are essential materials for students to learn mathematical concepts, including the rate of change. The way a textbook articulates an understanding of the rate of change affects how students understand the idea. Stein and colleagues (2007) indicate that standards-based textbook reformers focused on conceptual thinking, reasoning, and problem-solving and avoided focusing on drill of basic skills, which is often the focus of conventional textbooks. Because of the importance of rate of change in students' mathematical development and the proposed difference between standards-based and conventional textbooks, I focus this paper on differences and similarities between how rate of change is presented in one standards-based and one conventional textbook. In this paper, I respond to the following research question: What similarities and differences are there in how the one conventional and one standards-based (Algebra 1) textbook approach the rate of change related to three areas of ratio, proportion, and covariational reasoning?

Analytical Framework

Conceptual Analysis

Conceptual analysis is a method for describing how students might understand particular mathematical ideas in various ways (Thompson, 2008). I will use conceptual analysis to analyze how the idea of the rate of change is demonstrated for students' understanding conceptually in textbooks based on three important components (ratio, proportion, and covariational reasoning) of rate of change. Conceptual analysis can be used to establish models of students' knowledge and understanding at specific times and in specific situations to describe in four ways: "(1) in building models of what students actually know at some specific time and what they comprehend in specific situations, (2) in describing ways of knowing that might be propitious for students' mathematical learning, (3) in describing ways of knowing that might be deleterious to students' understanding of important ideas, and (4) in analyzing the coherence, or fit, of various ways of understanding a body of ideas" (Thompson, 2008, p. 45). In this paper, I use conceptual analysis as a basis to determine the coherence of various ways of comprehending rate of change where I situate my analysis in relation to three other key concepts ratio, proportion, and covariational reasoning.

Mathematical Meaning

Mathematical meaning for teaching focuses on fostering the connection among teachers' knowledge, how they teach, their contents, and what students learn (Thompson, 2013). Thompson (2013) addressed about absence of meaning in mathematics education by demonstrating that the preponderance of research on learning mathematics in the United States suggests that meaningful instruction is rare. Also, there is little expectation of developing students' mathematical meanings as well as using meanings in their thinking (Thompson, 2008). Textbooks can serve as a way to remediate the problems above by providing teachers and students with guidance in teaching and learning mathematical meanings.

Ratio, Proportion, and Covariational Reasoning

As a framework for doing my conceptual analysis of textbooks, I outline key understandings that, according to the research literature, support developing a robust understanding of rate of change. I chose to explore three components: ratio, proportion, and covariational reasoning as analytical lenses to explore the rate of change.

Ratio

There exists confusion regarding the distinction between ratio and rate in mathematics education research literature. Thompson (1994) stated three frequently discussed differences between ratio and rate that appear in the mathematics education literature: (a) a ratio is a multiplicative comparison of quantities that are of a similar kind (e.g., length compared to length) and a rate is a multiplicative comparison of two quantities that are different (e.g., height vs. time), (b) a ratio is a multiplicative comparison of one quantity in relation to another quantity and a rate is a ratio between a quantity and a period of time, and (3) a ratio is an ordered pair of quantities, and a rate is a relationship between one quantity and one unit of another quantity. He points out that these distinctions are based on situations rather than on the mental operations that enable students to establish mathematical meanings. He considers a characterization based on mental operations more helpful than one based on situations because the situation can vary depending on a mental operation by which one understands it. To be specific, when we take the perspective that ratios and rates are the products of mental operations, classification of situations between "rate" and "ratio" categories is no longer crucial (Thompson, 1994).

Related to the mental operations, Thompson (1994) defined a ratio as an outcome of a multiplicative comparison of two quantities. This definition implies two ways of comparison: (1) a comparison of two quantities as a whole, and (2) a comparison of one quantity related to the unit of another quantity. For instance, let there be two boys and five girls in a classroom. In the first comparison (1), $2 : 5$ will be the expression of two quantities as whole numbers. In the second comparison (2), however, $1 : 2\frac{1}{2}$ represents the number of girls relative to *one* boy. In other words, the number of boys has been assumed to be the unit per so the number of girls, $2\frac{1}{2}$, represents the ratio per boy. To sum up, definition (1) refers to *a ratio of whole numbers* and (2) refers to *a unit ratio*. In this paper, I will use the second comparison definition (2) of ratio as *a unit ratio* because the rate of change can be interpreted as a multiplicative comparison between the change in the *y* variable relative to the one unit of change in *x* variable. For example, if there are two points, $(1, 5)$ and $(3, 10)$, on the coordinate plane, the rate of change is that $\frac{\Delta y}{\Delta x} = \frac{10-5}{3-1} = \frac{5}{2}$. These can be expressed by $\Delta x : \Delta y \ = 2:5 \ or \ 1:\frac{5}{2}$. The latter one of $1:\frac{5}{2}$ shows a more advanced understanding of the rate of change than an expression of $2:5$, because *a unit ratio* enables students to establish equivalent ratios by multiplying the same numbers to *x* and *y* values, which is a coordination of the points in a linear line. In other words, in order to find a point on a linear line when *x* is 3, for the unit ratio of $1:\frac{5}{2}$, students can multiply 3 for both 1 and $\frac{5}{2}$, while for the ratio of $2:5$, it is harder to think of multiplying $\frac{3}{2}$ for both 2 and 5 to get $3:\frac{15}{2}$. Thus, I will use this definition (2) to compare one quantity related to the unit of another quantity in this paper.

Proportion

Understanding how to create equivalent ratios by iterating or partitioning a composed unit enables proportional reasoning (Lobato & Ellis, 2010). When the problem is given with a ramp that is 4 centimeters high and 5 centimeters long of the base to make all the ramps that have the same steepness as the original ramp but are not identical to it, students need to iterate or partition the initial ramp. For instance, firstly, students can make a copy of the original ramp, then both ramps have the same steepness because they have the same length of height and base. Secondly, students can align two ramps “tip to tip” will not change the steepness of either ramp. The result shows 8 centimeters high and 10 centimeters long base, which is twice the 4 and 5. Also, students can form smaller ramps by partitioning the original ramp. This essential understanding can lead to students’ knowledge of the slope of linear function but the partitioning and iterating the ramp are not enough to demonstrate the rate of change co-variationally as they do not address the relationship between base and height of ramp. To be specific, the essential understanding above is a lack of demonstration of how base and height are co-variationally related to the rate of change. Hence, in the next part, I want to discuss about how covariational reasoning helps understanding the rate of change.

Lobato and Ellis (2010) defined proportion as “a relationship of equality between two ratios (p.33).” The proportion is different from ratio because proportion implies equivalent values of the quantities, which can be created constantly. Without forming a ratio mentally, students have difficulty understanding the meaning of a proportion. Students may interpret the proportion structure of $\frac{a}{b} = \frac{c}{d}$ as a template for putting natural numbers into boxes (Lobato & Ellis, 2010). However, students without proportional reasoning cannot interpret the equation in this way. Proportional reasoning is the ability to establish the equality of two ratios to build up “a set of infinitely many equivalent ratios” which refers to rate (Lobato & Ellis, 2010).

In Algebra 1 textbooks, a proportion problem is often given to find one value, while three are given among four values such as $\frac{1}{4} = \frac{x}{8}$ in order to find one missing value passing through points by using the identical rate of change in linear function. To be specific, when a problem asks to find x, given a line passing through two points of $(2, 0)$ and $(10, x)$ with the slope of $\frac{1}{4}$, students can construct the equation of $\frac{1}{4} = \frac{x-0}{10-2} = \frac{x}{8}$, which is $\frac{1}{4} = \frac{x}{8}$. This kind of problem can be solved in two ways: (1) using multiplicative comparison and (2) cross multiplication formula. For example, the first way involves students’ multiplicative reasoning, which means $\frac{1}{4}$ is twice $\frac{1}{8}$, x is twice of $\frac{1}{8}$ so that x is 2. The second way involves memorization of the cross-multiplication procedure where $\frac{a}{b} = \frac{c}{d}$ means that $a \times d = b \times d$ so that students would make identical equation of $\frac{1}{4} = \frac{x}{8}$ as $1 \times 8 = 4 \times x$ and then find $x = 2$ by solving the equation. In this paper, I want to explore which textbook shows which ways of solving proportional problems when students work on finding a specific point of the linear line by using slope.

Covariational Reasoning

The rate of change can be expressed as a continuous covariation or discrete covariation between x and y values. Most of textbooks introduce the rate of change with tables with specific numbers that lead students to form discrete covariation rather than stretching to consider continuous covariation. Carlson and collegues (2002) investigated how to apply covariational reasoning with the framework related to the rate of change in college calculus. Carlson and collegues (2002) suggested an example of the Bottle Problem, “Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that’s in the bottle,” to illustrate common covariational reasoning (p. 360). This kind of problem enables for students to understand both continuous and discrete covariational reasoning. In this study, I want to explore how the textbooks instruct students to approach this kind of problems. Textbooks need to deliver not only discrete covariation but also continuous covariation in the concept of slope in a linear function.

Thompson and Carlson (2017) investigated variation, covariation, and functions in foundational ways of thinking mathematically. They introduced levels of variational and covariational reasoning to help researchers use it to describe a class of behaviors or to characterize a person's capacity to reason variationally or covariationally (Thompson & Carlson, 2017). In this paper, I am analyzing textbooks rather than students' work. Therefore, I modified the levels of covariation framework to examine textbooks.

Among six covariational reasoning levels, I chose the last five levels of precoordination of values, gross coordination of values, coordination of values, chunky continuous covariation, and smooth continuous covariation: (1) at the precoordination of values level, the textbook leads for students to envision two variables' values varying, but asynchronously – one variable changes, then the second variable changes, then the first, and so on while the person does not anticipate creating pairs of values as multiplicative objects, (2) at the gross coordination of values level, the textbook lets students form a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases," but the textbook does not envision that individual values of quantities go together, (3) at the coordination of values, the textbook leads students to coordinate the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y), (4) at the chunky continuous covariation level, the textbook lets students envision changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation, and (5) at smooth continuous covariation level, the textbook leads students to envision increases or decreases in on quantity's or variable's value as happening simultaneously with changes in another variable's value, and the textbook envisions both variables varying smoothly and continuously (Thompson & Carlson, 2017, p.435). The reason why I excluded the first level (no coordination) is that I assume every textbook will deliver coordination in some ways.

METHODS

Selected Textbooks

I selected two different types of Algebra 1 textbooks for analyzing the rate of change from linear function. Specifically, I chose one from conventional Algebra 1 textbooks and another one from standards-based books, which align with Algebra 1 contents of a linear function. My method for selecting the two textbooks involved using Stein et al. (2007) list of conventional and standards-based books. Then, I chose two textbooks from each category in the library. I checked whether the textbooks included lessons on rate of change in the linear function chapter and how they described and provided problems. I picked one textbook from each category that aligned well in terms of the amount of space dedicated to the topic of rate of change in the chapter on linear functions.

The conventional textbook I chose is *Algebra 1* published by McDougal-Littell (MLA) and the standards-based textbook is *Moving Straight Ahead* published by Connected Mathematics 3 (CMP). I wanted to compare or contrast how these textbooks reflect crucial components of rate of change in their treatment of linear functions. I analyzed specific portions of the textbooks to explore how two textbooks demonstrate introducing the rate of change. I tried to specify for the rate of change part and corresponding amount of the pages for each textbook in a limited time period. For MLA, I analyzed 28 pages of the parts of graphing linear equations and functions (pp. 234-261). For CMP, 29 pages of the part of exploring slope: connecting rates and ratios (pp. 87-112). Basically, I analyzed student textbooks while I checked the teacher versions of the textbooks for more understanding of the way of demonstrating the concepts within the solutions. Reformers for standards-based textbooks aimed to change textbooks from focusing on basic skills in conventional curriculum to focusing on conceptual thinking, reasoning, and problem solving (Stein et al., 2007). In this goal of standards-based textbook reformers, I assumed that I could see different depth of conceptual reasoning between a standards-based book and a conventional book.

Procedures and Coding

I produced codes based on using conceptual analysis of the rate of change. The components I want to use for coding are ratio, proportion, and covariational reasoning based on the analytical framework. To make codes, I will define the criteria of each component to be clear for coding. Also, I will revise the coding with examples or additional criteria as I analyze the textbooks. In this paper, I want to explore which textbook shows which ways of solving the rate of change problems when students work on the linear line.

For the procedures of the analysis, I tried to set coding table to analyze the textbook based on the codes. Also, I coded two textbooks by using MAXQDA, which is qualitative analysis software program. As I determined to code in three different categories of ratio, proportion, and co-variation, I decided criteria for each sub-topic within three categories of ratio, proportion, and co-variation. In order to find meaningful coding criteria, I brought the coding scheme from analytical framework through literature review and applied them to analyze textbooks. I tried to keep checking textbook materials to compare and contrast the two types of textbooks, conventional and standards-based.

Table 1
Coding scheme for textbook analysis

Topic	Criteria	Codes
Ratio	Comparison of two quantities as a whole Ex. If two points of (0,0) and (2,3) are given to find slope, students can find the slope with whole number expression of $\frac{3}{2}$ by considering the ratio of whole number difference of y-value, 3-0=3, and x-value, 2-0=2.	R1
	Comparison of one quantity related to the unit of another quantity Ex. If two points of (0,0) and (2,3) are given to find slope, students can find the slope of $\frac{3}{2}$ as well as conceive the slope as the ratio of $1{:}\frac{3}{2}$.	R2
Proportion	Iterating or partitioning a composed unit to create equivalent ratios Ex. Find equivalence of $\frac{1}{4}=\frac{2}{8}$ by iterating one twice to make two and iterating four twice to make 8. Or, conceive the correspondence by partitioning 8 in half to create four and partitioning 2 in half to make 1.	P1
	Multiplicative comparison to find unknown number in a proportion form Ex. While students solve the rate of change problem such as finding x, given a line passing through two points of $(2,0)$ and $(10,x)$ with the slope of $\frac{1}{4}$, the proportional form of $\frac{1}{4}=\frac{x}{8}$ can be made and then students can find x by using multiplicative comparison that 8 is twice 4, x is twice of 1 so that x is 2.	P2
	Constant equality of two ratios to build up a set of infinitely many equivalent ratios. Ex. $\frac{1}{4}=\frac{2}{8}=\frac{3}{12}=\cdots$	P3

Covariational Reasoning	Precoordination of values Envision two variables' values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. Not anticipate creating pairs of values as multiplicative objects. Ex. In a bottle problem, textbook describes that after some amount of water is poured into the bottle, the water level on the bottle rises.	C1
	Gross Coordination of values Form a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." Envision a loose, nonmultiplicative link between the overall changes in two quantities' values, but not envision that individual values of quantities go together. Ex. In a bottle problem, textbook describes the covariation as "the height increases as the volume increases."	C2
	Coordination of values Coordinate the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y). Ex. In a bottle problem, textbook focuses on the water's height in the bottle and the number of cups of water added to the bottle with no thought given to intermediate values of volume or height.	C3
	Chunky continuous covariation Envision changes in one variable's value as happening simultaneously with changes in another variable's value and envision both variables varying with chunky continuous variation. Ex. In a bottle problem, textbook describes the water level rising for each increment of water added, including all values of volume and height between successive values, but without envisioning height and volume passing through those values.	C4
	Smooth continuous covariation Envision increases or decreases in one quantity's or variable's value as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously. Ex. In a bottle problem, textbook describes both the water's volume and height varying smoothly through intervals simultaneously, while anticipating that within each interval the amount of water and height of water vary smoothly and continuously.	C5

My coding scheme (**Table 1**) included criteria of key components from ratio, proportion, and covariational reasoning related to the rate of change.

FINDINGS

I use **Table 2** to report the inclusion of the various components of ratio, proportion, and covariational reasoning. Analyzing two textbooks with these criteria, I recorded which book shows which components from the three categories. **Table 2** indicates the trend within and across the two textbooks. Code result shows comparison of two textbooks of MLA and CMP in **Table 2**. Generally, for the similarities, the result demonstrates that both textbooks contain R1, R2, P2, C1, C2, C3, C4 and C5, while MLA includes P3 and CMP does not. On the other hand, the CMP textbook contains the components of P1 while MLA does not.

Table 2
Methods for exploring ratio, proportion, and fraction related to the rate of change

		MLA	CMP
Ratio	R1	√	√
	R2	√	√
Proportion	P1		√
	P2	√	√
	P3	√	
Covariational Reasoning	C1	√	√
	C2	√	√
	C3	√	√
	C4	√	√
	C5	√	√

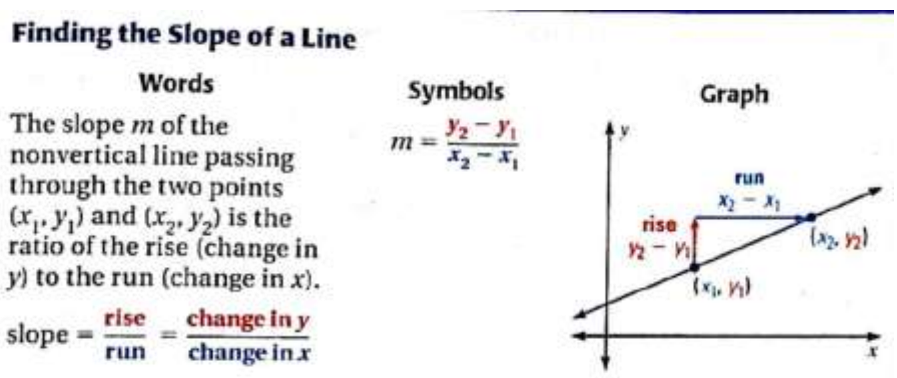
Ratio

Comparison of Two Quantities as A Whole (R1)

Both textbooks showed Comparison of two quantities as a whole (R1) with ratio of the rise to the run. It was established in the same order of the rise first and then run. Also, both MLA and CMP books tried for students to learn how to find the slope by using the ratio of the rise to the run.

Figure 1
Comparison of two quantities as a whole (R1)

(a) MLA



(*Larson et al., 2010, p.235)*

(b) CMP

B A set of stairs is being built for the front of the new Arch Middle School. The ratio of rise to run is 3 to 5.

(Lappan et al., 2014, p.89)

The MLA textbook formulates how to find the slope by using the ratio of the rise to the run in three ways of words, symbols, and graph (**Figure 1a**). Two lengths of the rise and run are the objects to be compared. Likewise, CMP explained the rate of change by using ratio of two quantities, which have the same standard of length. The textbook described the rate of change as a ratio of the rise to the run phrasing. In the question (**Figure 1b**), the ratio was given, and let students explain and figure each stair's length of the rise and the run by exploring more within an activity.

The MLA textbook introduced the slope with its formula at the beginning of the chapter and then examples to practice substituting corresponding numbers to the formula. The real-world problems are given after practicing the formula of finding slope with examples. For the CMP textbook, the contextual question came on the first page of the chapter. There is no stated formula throughout the textbook but students need to generalize and find how to find slope in their own ways by exploring given word problems. Both textbooks showed the same order that rise first and then run with the same shape of the right triangle with a hypothesis at the bottom. In other words, they introduced the slope with the positive slope line as a hypothesis of a right triangle.

Comparison of One Quantity Related To the Unit of another Quantity (R2)

The MLA and CMP included comparing one quantity related to the unit of another quantity (R2) in the rate of change. Having the notion of the quantity compared to the unit of another quantity enables students to conceive the relationship between two quantities.

Figure 2

Comparison of one quantity related to the unit of another quantity (R2)

(a) MLA

Solution

$$\text{Rate of change} = \frac{\text{change in cost}}{\text{change in time}} = \frac{14-7}{4-2} = \frac{7}{2} = 3.5$$

▸ The rate of change in cost is $3.50 per hour.

(*Larson et al., 2010, p.237)*

(b) CMP

7.RP.A.2d Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r), where r is the unit rate.

(Lappan et al., 2014, p.87)

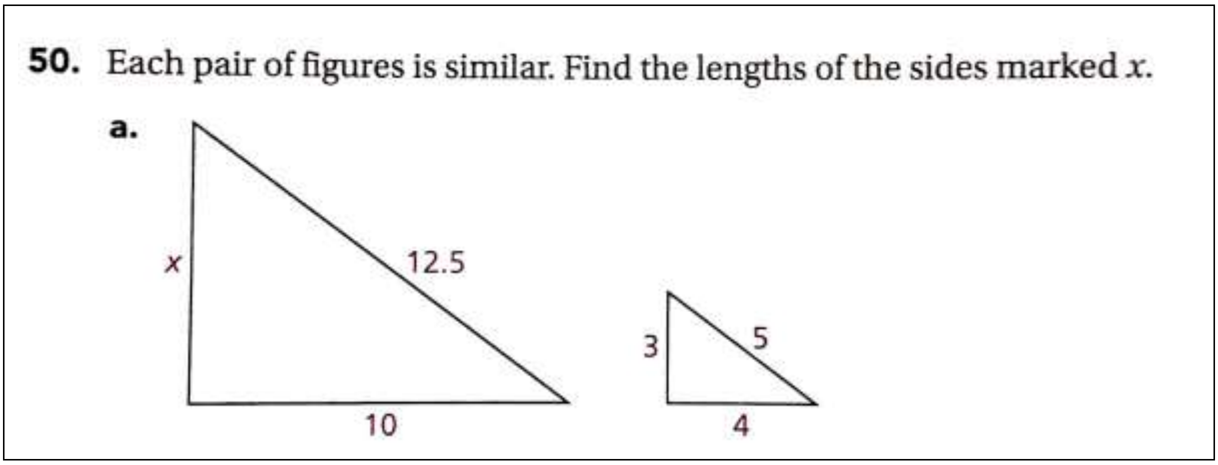
The MLA textbook demonstrated a rate of change by comparing one quantity related to the unit of another quantity (**Figure 2a**). For example, in the description, it leads the rate of change from $60 per 5 hours to $12 per hour. It implies a trial to change unit from 5 to 1 in order to see the rate of change per 1 hour unit rather than per 5-hour unit. In the same way, the solution tried to reason of $\frac{7}{2} = \frac{3.5}{1}$, which can be explained by changing from $7 per 2 hours to $3.5 per hour, which includes the concept of comparison of one quantity related to the unit of another quantity. On the other hand, the CMP textbook focuses on delivering the unit rate, which can be the rate of y-value compared to the unit 1 value of x. This is also can help students to understand the length of the rise compared to 1 unit of the run.

Proportion

Iterating or Partitioning a Composed Unit to Create Equivalent Ratios (P1)

CMP only included iterating or partitioning a composed unit to create equivalent ratios (P1) while MLA did not include the code (P1).

Figure 3
Iterating or partitioning a composed unit to create equivalent ratios (P1) (Lappan et al., 2014, p.110)



In **Figure 3**, smaller ramp's base is 4 and height 3 while bigger ramp's base is 10 and height is unknown x. Students can partition 4 to 2, which is half of four, and then iterate two five times to make 10. As two ramps are similar given by the problem, students can partition smaller ramp's height 3 to 1.5, which is half of the three, and then iterate five times of 1.5 to make 7.5, which is unknown x. In this problem, students can create equivalent ratio of height to base as $\frac{3}{4}$ and $\frac{x}{10}$ because similar figures have the identical corresponding angles and the same slope for each line.

Multiplicative Comparison to Find Unknown Number in A Proportion Form (P2)

Both MLA and CMP textbooks included multiplicative comparison to find unknown number in a proportion form (P2). As students learned the proportion in their previous grades of 4-6, this is a revision for students. I wanted to explore how textbooks revisit proportion in the context of the rate of change (slope).

The MLA suggested the problems of proportion by using various ways of a combination of numbers and unknowns (**Figure 4a**). For example, while the number 55 problem shows the basic proportion form, 56 and 57 are more complicated to solve problems because they have equations with unknowns in both the numerator (2x) and the denominator (x+4). In the basic form of #55, students can find x by thinking of 50 is ten times of five so x will be ten times four, which is 40. However, it could be hard to use this reasoning on problems #56 and #57 as the problems contain unknown equations rather than specific numbers. In the teachers' book, I can see the textbook delivers the formula to solve proportion form problems by multiplying across and having them as the same. I argue that the MLA book emphasized the complicated computation skills than how to reason the proportion form with their multiplicative concept.

The CMP book included the proportion form of questions (**Figure 4b**). The questions a, b, c, and d of #51 can be solved by using multiplicative concepts. For example, in #51.a, 10 is five times two so n is five times three, which is 15. Among the problems, #51.c is the most challenging because it is a negative sign and 3 is half of the six. In other words, the problem requests to think of fractional knowledge and integer characteristics of negative signs. Finally, I focused on the #51.e because the problem intended to connect the proportion form to the rate of change (slope). It is impressive to let students think of the relationship between computational problems of proportion form and the concept of the rate of change. I argue that the CMP book emphasized the connection that needs mathematical reasoning than intensive computational skills only.

Figure 4
Multiplicative comparison to find unknown number in a proportion form (P2)

(a) MLA

Solve the proportion. Check your solution. *(p. 168)*

55. $\frac{4}{5} = \frac{x}{50}$ **56.** $\frac{2x}{x+4} = \frac{8}{9}$ **57.** $\frac{7t-2}{8} = \frac{3t-4}{5}$

(*Larson et al., 2010, p.250)*

(b) CMP

51. Find a value of n that will make each statement true.

a. $\frac{n}{10} = \frac{3}{2}$ **b.** $\frac{5}{6} = \frac{n}{18}$ **c.** $-\frac{4}{6} = \frac{n}{3}$ **d.** $\frac{5}{18} = \frac{20}{n}$

e. Write an equation for a line whose slope is $-\frac{4}{6}$.

(Lappan et al., 2014, p.87)

Constant Equality of Two Ratios to Build Up a Set of Infinitely Many Equivalent Ratios (P3)

Only MLA textbook included the constant equality of two ratios to build up a set of infinitely many equivalent ratios (P3) while CMP does not include it. The linear line function has a constant slope which is the same for every point on the linear line.

Figure 5
Constant equality of two ratios to build up a set of infinitely many equivalent ratios (P3) (*Larson et al., 2010, p.256)*

ONLINE MUSIC The table shows the cost C of downloading s songs at an Internet music site.

a. Explain why C varies directly with s.

b. Write a direct variation equation that relates s and C.

Number of songs, s	Cost, C (dollars)
3	2.97
5	4.95
7	6.93

Solution

a. To explain why C varies directly with s, compare the ratios $\frac{C}{s}$ for all data pairs (s, C): $\frac{2.97}{3} = \frac{4.95}{5} = \frac{6.93}{7} = 0.99$.
Because the ratios all equal 0.99, C varies directly with s.

b. A direct variation equation is $C = 0.99s$.

In **Figure 5**, the solution sets the constant equality of two ratios based on the table and the clue of a direct variation equation in the problem. Through the question, students can confirm whether the slope (the rate of change) is identical for all points they have when the function is linear (direct variation equation). Furthermore, students are able to form linear equations with the slope they found in the previous sub-problem. This problem solving enables students to understand the meaning of slope with multiple points on it.

Covariational Reasoning

Precoordination of Values (c1)

Figure 6

Precoordination of values (C1)

(a) MLA

44. ★ **EXTENDED RESPONSE** An artist is renting a booth at an art show. A small booth costs $350 to rent. The artist plans to sell framed pictures $50 each. The profit P (in dollars) the artist makes after selling p pictu is given by $P = 50p - 350$.

a. Graph the equation.

(*Larson et al., 2010, p.249*)

(b) CMP

After five weeks, I will have saved a total of $175

After eight weeks, I will have saved $190.

1. How much of her allowance is Chantal planning to save each week?

2. How much birthday money did Chantal's grandfather give her?

(Lappan et al., 2014, p.95)

Gross Coordination of Values (C2)

In **Figure 7**, both textbooks include the gross coordination of values (C2) components in a way that delivers a concept of the rate of change. The C2 is embedded in the problems of negative sign slope as x is increasing while y is decreasing in negative sign slope.

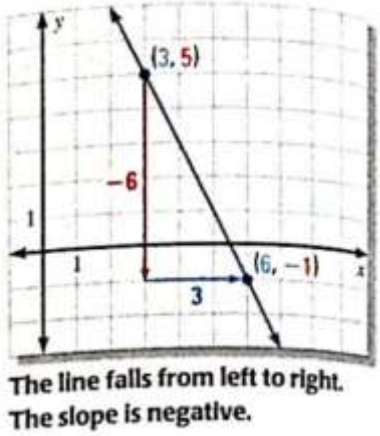
The MLA reveals the decreasing of six-units y-value by having red color arrow from top to bottom (from 5 to -1) while x-value is increasing three-units x-value from left to right (from 3 to 6) with the blue color arrow. This visual assistance such as arrow and color enable students to understand the order and pair of each group and form the process. CMP textbook also included C2 with the negative sign slope problem. However, the problems do not convey the process explicitly. I infer that the CMP textbook intends for students to find the slope by exploring such problems rather than providing a solution-oriented expression on it.

Coordination of Values (C3)

Both MLA and CMP textbooks included the coordination of values (C3) by stating the corresponding x-value and y-value as pairs. The coordinating numbers come with the graph in order to show the corresponding numbers from the graph. However, this does not mean the textbook supports the relationship between x and y as independent and dependent variables.

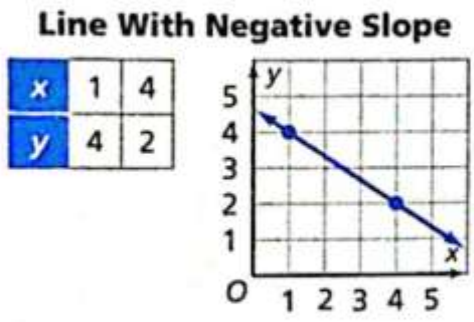
Figure 7
Gross coordination of values (C2)

(a) MLA



(*Larson et al., 2010, p.236)*

(b) CMP

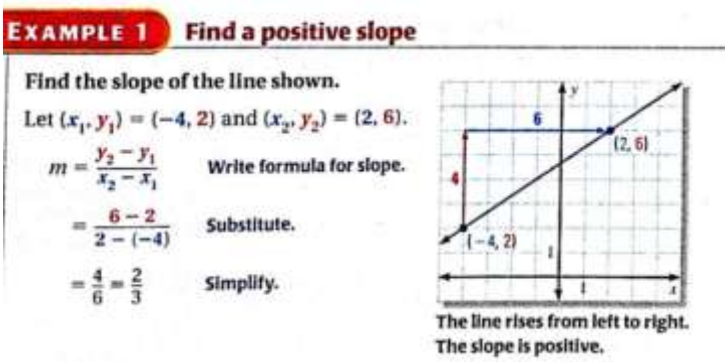


(Lappan et al., 2014, p.91)

In the MLA textbook (**Figure 8a**), the example highlighted with blue color for x-values and red for y-values in the first line as $(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (2, 6)$. This coloring expression supports students in matching the coordinating numbers correctly to the x-values and y-values, respectively. By contrast, the CMP textbook (**Figure 8b**) does not include explicit examples to demonstrate the coordination of values. CMP questioned about meaning of constant rate of change related to the relationship between independent and dependent variables. To sum up, I argue MLA textbook included explicit examples without meaning of the relationship between independent and dependent variables related to the rate of change while CMP questioned students' reasoning for the constant rate of change.

Figure 8
Coordination of values (C3)

(a) MLA



(*Larson et al., 2010, p.235)*

(b) CMP

2. In previous Investigations, you learned that linear relationships have a constant rate of change. As the independent variable changes by a constant amount, the dependent variable also changes by a constant amount. How is the constant rate of change of a linear relationship related to the slope of the line that represents that relationship?

(Lappan et al., 2014, p.93)

Chunky Continuous Covariation (C4)

Both MLA and CMP textbooks included chunky continuous covariation (C4) code.

In **Figure 9**, MLA textbook asks the greatest or the least time interval as a chunk of the constant slope.

CMP textbook reminds for students to consider two stairs at a time to identify the slope compared to one stair's slope. Also, the textbook leads students to check the slope every stair to check constant slope in the linear line of stairs.

Figure 9
Chunky continuous covariation (C4)

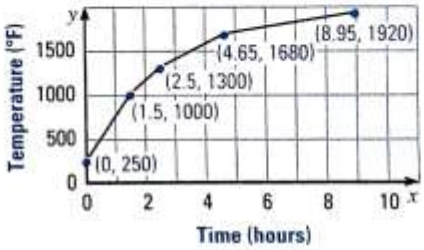
(a) MLA

37. **MULTI-STEP PROBLEM** Firing a piece of pottery in a kiln takes place at different temperatures for different amounts of time. The graph shows the temperatures in a kiln while firing a piece of pottery (after the kiln is preheated to 250°F).

a. Determine the time interval during which the temperature in the kiln showed the greatest rate of change.

b. Determine the time interval during which the temperature in the kiln showed the least rate of change.

@HomeTutor for problem solving help at classzone.com



(Larson et al., 2010, p.241)

(b) CMP

The steepness of the line is the ratio of rise to run, or vertical change to horizontal change, for this step. We call this ratio the **slope** of the line.

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

- Does the slope change if we take two stairs at a time?
- Is the slope the same between any two stairs?

(Lappan et al., 2014, p.90)

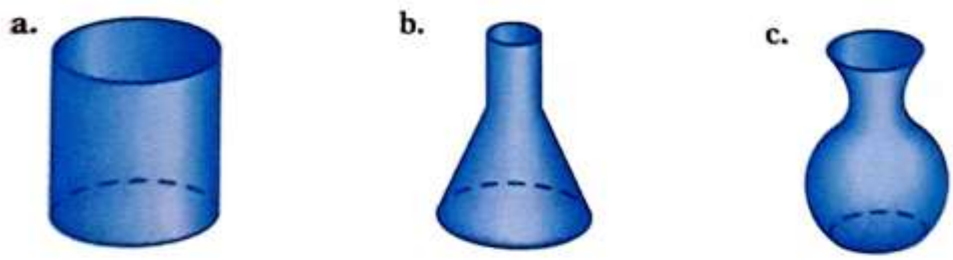
Smooth Continuous Covariation (C5)

Both MLA and CMP textbooks included smooth continuous covariation (C5) code implicitly. In MLA textbook (**Figure 10a**), the problem implies time and pouring water happen simultaneously and those variables varying smoothly and continuously. In CMP textbook, the problem 10.b included the question about the meaning of the slope in the situation. This question implies the possibility of reaching out to smooth continuous covariational reasoning. However, both MLA and CMP problems did not include how to deliver these problems to students or solution which show the evidence for coding C5 in both students' textbooks and teachers' edition textbooks. Thus, I concluded C5 code are shown implicitly in both textbooks.

Figure 10
Smooth continuous covariation (C5)

(a) MLA

41. CHALLENGE Imagine the containers below being filled with water at a constant rate. Sketch a graph that shows the water level for each container during the time it takes to fill the container with water.



(*Larson et al., 2010, p.242)*

(b) CMP

37. At noon, the temperature is 30°F. For the next several hours, the temperature falls by an average of 3°F an hour.

a. Write an equation for the temperature T, n hours after noon.

b. What is the y-intercept of the line the equation represents? What does the y-intercept tell you about this situation?

c. What is the slope of the line the equation represents? What does the slope tell you about this situation?

(Lappan et al., 2014, p.105)

CONCLUSION

The research question was: what similarities and differences are there in how the two conventional and standards-based (Algebra 1) textbooks approach the rate of change related to four areas of ratio, proportion, and covariational reasoning?

Similarities

The MLA and CMP textbooks include R1, R2, P2, C1, C2, C3, C4, and C5 in common. For the ratio, the MLA textbook delivered the comparison of two quantities as a whole (R1) and a comparison of one quantity related to the unit of another quantity (R2) with examples including solutions in the student textbook. The CMP textbook included R2 with the contextual problems without solutions in the student textbook, so students need to explore how to solve the problems by themselves. Multiplicative comparison to find unknown number in a proportion form (P2) is conveyed by MLA and CMP to find a variable from four numbers by using the same slope proportion. However, MLA focused on complicated computational skills while CMP shows focusing on the contextual meaning of the identical two proportions. As the rate of change of linear function includes coordinating pairs of points with both discrete and continuous covariation, precoordination of values (C1), gross coordination of values (C2), coordination of values (C3), chunky continuous covariation (C4), and smooth continuous covariation (C5) are included in both textbooks.

Differences

The MLA included constant equality of two ratios (P3) while CMP did not have it. MLA demonstrated P3 by having examples with solutions that show multiple equivalences from the pairs in the **Table 1** expected students to find equivalent constant ratios, but the textbook let students find the slope from the different pairs of ratios. On the other hand, the CMP included iterating or partitioning a composed unit to create equivalent ratios (P1), but MLA did not contain it.

To sum up, I argue that both textbooks contain most of the components (R1, R2, P2, C1, C2, C3, C4 and C5) from three sections ratio, proportion, and covariational reasoning. MLA textbook indicates focusing on computational skills by having solutions embedded in concise examples to introduce procedures for solving problems. By contrast, the CMP textbook shows the trend of focusing on students' reasoning and mathematical thinking by providing contextual problems that let students explore and find their own way of solutions through the procedure. This can be more challenging for students to learn how to solve math problems as there are fewer guidelines than MLA.

Ethical Statement

The research presented in this paper involves a conceptual analysis and comparison of publicly available, commercially published Algebra 1 textbooks. This study did not involve the collection of data from human participants, live subjects, or identifiable personal information. Therefore, formal ethical approval, consent to participate, or consent for publication was not required.

Ethics Committee Approval

Not applicable

Author Contributions

The corresponding author is the sole author of this manuscript and is responsible for all aspects of the research, conceptualization, methodology, analysis, writing, and submission.

Finance

Not applicable

Conflict of Interest

Not applicable

REFERENCES

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Lappan, G., Phillips, E. D., Fey, J. T., & Friel, S. N. (2014). Moving straight ahead. In *Connected mathematics 3*.

Larson, R., Boswell, L., Kanold, T. D., & Stiff, L. (2010). 4. Graphing linear equations and functions. In *McDougal Littell Algebra 1*.

Lobato, J., & Ellis, A. B. (2010). *Developing essential understanding of ratios, proportions & proportional reasoning for teaching mathematics in grades 6-8* (R. I. C. R. M. Z. Mary (ed.)). National Council of Teachers of Mathematics.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In *Second handbook of research on mathematics teaching and learning* (pp. 319–369). National Council of Teachers of Mathematics, Inc. https://doi.org/10.1016/j.ece.2017.03.001

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). https://doi.org/10.1016/b978-0-08-057109-6.50014-8

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). Morelia: PME.

Thompson, P. W. (2013). In the Absence of Meaning... In *Vital directions for research in mathematics education* (Vol. 8, Issue 2, pp. 57–93). https://doi.org/10.2752/155280105778055254

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of mathematical thinking. *Compendium for Research in Mathematics Education*, *January*, 421–456.